Grad's Moment Description of a Heterogeneous Dispersed Media in the Frame of the Standard Enskog Theory

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Abstract. Two-phase polydispersed media appear in a wide range of engineering process and environmental problems. Therefore, the identification of equations allowing a good prediction of the behaviour of such media is an important challenge. The aim of this paper is to provide an Eulerian description for a heterogeneous suspension constituted of different solid particle species. Kinetic equations for such media have been written in previous papers [1, 2, 3] with Grad's thirteen moments method [4] to obtain an approximation of the transport equation collision integral in the frame of the Standard Enskog Theory. In this work, we are interested in the case where each particle species has its own temperature which is close to the equilibrium temperature.

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INTRODUCTION

Because of the growing industrial importance of micro and nano technologies, the human sources of production of solid nano particles increase. These particles are used, for instance, in cosmetic and pharmaceutical industry as vector for active substances or in car tyre industry to improve their performance. The increasing presence, in the environment, of these extremely small objects (of size lower to 100 nm), asks the question of the valuation of health risks that they can cause by penetrating into organism. Consequently, the study of gaseous flows loaded with solid nano particles and the identification of equations allowing to predict their behaviour are very important. In this paper, we provide a kinetic description for a heterogeneous suspension constituted of N different particle species and carried by a viscous gas. In [3], an Eulerian description for the dispersed media has been obtained in the case where the "temperature" of each species is equal to the equilibrium temperature. In this work, we present a more general set of equations with a different "temperature" for each species: We search each species "temperature" as an asymptotical expansion around the equilibrium temperature. In the next section we present the main assumptions about the suspension and the collisional process and we recall, briefly, the main results on the kinetic description of a suspension of heterogeneous spheres having instantaneous, binary and inelastic collisions [1]. In the following sections, as in the usual kinetic theory, the method of Grad's thirteen moments [4, 5, 6] is used to obtain an approximation of the collision integral of the transport equation. Thus, macroscopic conservation equations for each species and global balance equations are introduced in the frame of the Standard Enskog Theory. The last section is devoted to a short conclusion.

KINETIC DESCRIPTION OF THE SUSPENSION

In this work, the carrier fluid is assumed to be a viscous gas and the collisions between particles are supposed to be instantaneous, binary, weakly inelastic and non punctual. The suspension is assumed to be heterogenous and to be made up of N particle species. A particle α ($\alpha = 1, ..., N$) is assumed to be a sphere of diameter σ_α and of mass m_α. It is centered at position $\mathbf{x}_\alpha$ and has the velocity $\mathbf{v}_\alpha$. First, the collision of two spheres P_α and P_β is considered. We put $\sigma_{\alpha\beta} = (\sigma_\alpha + \sigma_\beta)/2$ and we suppose that, before the collision the particles P_α and P_β are centered at position $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ and have the velocities $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$. The relative velocity $\mathbf{g}_{\beta\alpha}$ and the impact vector $\mathbf{k}$ are defined by $\mathbf{g}_{\beta\alpha} = \mathbf{v}_\beta - \mathbf{v}_\alpha$ and $\mathbf{k} = (\mathbf{x}_\alpha - \mathbf{x}_\beta)/ \mid \mathbf{x}_\alpha - \mathbf{x}_\beta \mid$ with $\mathbf{x}_\beta = \mathbf{x}_\alpha + \sigma_{\alpha\beta}\mathbf{k}$ during a collision. After the collision, the particles are still centered in $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ and they have the velocities $\mathbf{v}'_\alpha$ and $\mathbf{v}'_\beta$. The relative velocity after collision is

$\mathbf{g}^{'}_{\beta\alpha} = \mathbf{v}^{'}_{\beta} - \mathbf{v}^{'}_{\alpha}$. Furthermore, because of the inelasticity of the collisions, we introduce the coefficient of restitution $e_{\alpha\beta}$ with the relation $\mathbf{g'}_{\beta\alpha} \cdot \mathbf{k} = -e_{\alpha\beta} \left(\mathbf{g}_{\beta\alpha} \cdot \mathbf{k} \right)$ with $0 < e_{\alpha\beta} = e_{\beta\alpha} \leq 1$. From experimental works [7], in the case of coal powder, the collisions are weakly inelastic and the restitution coefficient is $e_{\alpha\beta} \simeq 0,95$. If $e_{\alpha\beta} = 1$, the collisions are elastic and energy is conserved during the collisions. By considering the momentum balance equation and the previous assumptions, it's easy to express the velocities after collision in terms of those before:

$$\mathbf{v}^{'}_{\alpha} = \mathbf{v}_{\alpha} + \frac{(1+e_{\alpha\beta})\, m_{\beta}}{m_{\alpha}+m_{\beta}} \left(\mathbf{g}_{\beta\alpha} \cdot \mathbf{k} \right) \mathbf{k} \qquad \text{and} \qquad \mathbf{v}^{'}_{\beta} = \mathbf{v}_{\beta} - \frac{(1+e_{\alpha\beta})\, m_{\alpha}}{m_{\alpha}+m_{\beta}} \left(\mathbf{g}_{\beta\alpha} \cdot \mathbf{k} \right) \mathbf{k} \tag{1}$$

and vice-versa. In order to provide a kinetic description for a heterogeneous suspension constituted of N different solid particle species, it is convenient to introduce a distribution function $f_{\alpha}(\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, t)$ for each species and to write a Boltzmann equation for each of these N distribution functions. We recall the Boltzmann equation introduced in a previous work [1] for the species α:

$$\frac{\partial f_{\alpha}}{\partial t} (\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, t) \; + \; \mathbf{v}_{\alpha} \cdot \frac{\partial f_{\alpha}}{\partial \mathbf{x}_{\alpha}} (\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, t) \; + \; \frac{\partial}{\partial \mathbf{v}_{\alpha}} \cdot \left[\mathbf{F}_{\alpha} f_{\alpha} (\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, t) \right] = \sum_{\beta=1}^{N} J_{\alpha\beta} \left[f_{\alpha} (\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, t) \, , f_{\beta} \left(\mathbf{v}_{\beta}, \mathbf{x}_{\beta}, t \right) \right] \tag{2}$$

where $J_{\alpha\beta}(f_{\alpha}\, , f_{\beta})$ is the part of the collision operator associated with the collisions between the particles α and β and $\mathbf{F}_{\alpha}$ is the external force per unit of mass acting on a particle α. This force depends on the velocities and on the nature of the fluid and of the particle. In order to express the collision integral, we introduce the distribution function $f^{(2)}_{\alpha\beta}(\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, \mathbf{v}_{\beta}, \mathbf{x}_{\beta}, t)$ which caracterizes the statistic of binary collisions and depends on the velocities and positions of two particles α and β and time. It is defined so that $f^{(2)}_{\alpha\beta}(\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, \mathbf{v}_{\beta}, \mathbf{x}_{\beta}, t)\, d\mathbf{v}_{\alpha}\, d\mathbf{v}_{\beta}\, d\mathbf{x}_{\alpha}\, d\mathbf{x}_{\beta}$ is the probable number of pairs of particles which are, at time t, in $(\mathbf{x}_{\alpha}, d\mathbf{x}_{\alpha})$ and $(\mathbf{x}_{\beta}, d\mathbf{x}_{\beta})$ with velocities respectively in $(\mathbf{v}_{\alpha}, d\mathbf{v}_{\alpha})$ and $(\mathbf{v}_{\beta}, d\mathbf{v}_{\beta})$.We introduce also the pseudo-inverse collision [6, 8, 9], that is, the collision during which the two particles P_{α} and P_{β} collide with the velocities $\mathbf{v}^{''}_{\alpha}$ and $\mathbf{v}^{''}_{\beta}$ before collision and $\mathbf{v}_{\alpha}$ and $\mathbf{v}_{\beta}$ after collision. Consequently, the relations (1) can be used to express $\mathbf{v}_{\alpha}$ and $\mathbf{v}_{\beta}$ in terms of $\mathbf{v}^{''}_{\alpha}$ and $\mathbf{v}^{''}_{\beta}$. Under these assumptions, we have:

$$J_{\alpha\beta}(\mathbf{x}_{\alpha}, t) = \frac{\sigma^2_{\alpha\beta}}{e^2_{\alpha\beta}} \int_{(\mathbf{g}_{\beta\alpha} \cdot \mathbf{k})>0} \left[f^{(2)}_{\alpha\beta}(\mathbf{v}^{''}_{\alpha}, \mathbf{x}_{\alpha}, \mathbf{v}^{''}_{\beta}, \mathbf{x}_{\alpha} + \sigma_{\alpha\beta}\mathbf{k}, t) - e^2_{\alpha\beta} f^{(2)}_{\alpha\beta}(\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, \mathbf{v}_{\beta}, \mathbf{x}_{\alpha} - \sigma_{\alpha\beta}\mathbf{k}, t) \right] (\mathbf{g}_{\beta\alpha} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{v}_{\beta} \tag{3}$$

In the case of a heterogeneous suspension, we set [10]:

$$f^{(2)}_{\alpha\beta}(\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, \mathbf{v}_{\beta}, \mathbf{x}_{\beta}, t) = \chi_{\alpha\beta}(\mathbf{x}_{\alpha}, \mathbf{x}_{\beta})\, f_{\alpha}(\mathbf{v}_{\alpha}, \mathbf{x}_{\alpha}, t)\, f_{\beta}(\mathbf{v}_{\beta}, \mathbf{x}_{\beta}, t) \tag{4}$$

where $\chi_{\alpha\beta}$ is the radial distribution function that takes into account that the hard particles cannot penetrate each other. The expression of this distribution function has been discussed in several papers [10, 11, 12, 13, 14]. In the Standard Enskog Theory (SET), $\chi_{\alpha\beta}$ is the local equilibrium value of the radial distribution function evaluated as a function of the local densities n_{δ} ($\delta = 1, .., N$) at some particular point $\mathbf{X}_{\alpha\beta}$. It is the same function as in a fluid mixture in uniform equilibrium [13]. Different choices are possible for this point: The point of contact of the two colliding particles, the midpoint of the line connecting their centers or their center of mass for instance. Obviously, in the case of a monodispersed suspension, these three choices coincide but, when the particles P_{α} and P_{β} are different, each choice leads to a different solution of the kinetic equation and, obviously, to a different set of macroscopic balance equations. In [12], a revision of the Enskog theory is proposed in order to avoid this matter: $\chi_{\alpha\beta}$ has to be taken as the a non local functional of the density fields of the components of the heterogeneous medium. These functionnals are the same as in a fluid mixture in nonuniform equilibrium. Nevertheless, the extension of this gas theory to solid particles requires the generalization thermodynamical quantities such as the chemical potential. Because of the physical differences between gas molecules and solid particles, this generalization is not easy. Moreover, in this paper, we are interested in a macroscopic description at the same accuracy level as the model presented in [5] for homogeneous dispersed media. As in [3], we will show in the following that, at this level of description, it is convenient to introduce a SET model.

GLOBAL BALANCE EQUATIONS

The case of homogeneous suspensions has been studied in several papers [5, 6, 8, 9, 14, 15]. In [5, 6, 8], a macroscopic description by the method of Grad's thirteen moments has been given for the dispersed media. In the heterogeneous case, we define the mean value $<\gamma>_\alpha$ of a given particle property γ by the following expression:

$$<\gamma>_\alpha = \frac{1}{n_\alpha} \int \gamma f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)\, d\mathbf{v}_\alpha \tag{5}$$

The velocity fluctuation $\mathbf{c}_\alpha$ is defined so that $\mathbf{c}_\alpha = \mathbf{v}_\alpha - \mathbf{u}_\alpha$ where $\mathbf{u}_\alpha =< \mathbf{v}_\alpha >_\alpha$ is the mean velocity of the species α. As in the usual kinetic theory [10, 16], the Boltzmann equation (2) is multiplied by a function $\psi_\alpha(\mathbf{c}_\alpha)$ and integrated on the $\mathbf{v}_\alpha$ velocity space in order to obtain a transport equation:

$$\int \left\{ \frac{\partial f_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}_\alpha} + \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot [\mathbf{F}_\alpha f_\alpha] \right\} \psi_\alpha(\mathbf{c}_\alpha) d\mathbf{v}_\alpha = \sum_{\beta=1}^{N} \int \left\{ J_{\alpha\beta}\left[f_\alpha , f_\beta\right] \right\} \psi_\alpha(\mathbf{c}_\alpha) d\mathbf{v}_\alpha = \sum_{\beta=1}^{N} C_{\alpha\beta}(\psi_\alpha) \tag{6}$$

The first member of this equation is obvious to express. This has been done in [1, 2] for instance. Concerning the right hand side member of the transport equation, after a few technical steps (change of notations, use of the pseudo-inverse collision, exchange between particles P_α and P_β), the following equality is written [1]:

$$C_{\alpha\beta}(\psi_\alpha) = \sigma_{\alpha\beta}^2 \int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \Big(\psi_\alpha(\mathbf{c}_\alpha') - \psi_\alpha(\mathbf{c}_\alpha) \Big) f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x}_\alpha, \mathbf{v}_\beta, \mathbf{x}_\alpha - \sigma_{\alpha\beta}\mathbf{k}, t)(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\ d\mathbf{k}\, d\mathbf{v}_\beta\ d\mathbf{v}_\alpha \tag{7}$$

The previous expression of the collision integral can be easily used to prove exactly the conservation of mass for each species. An additionnal step of symmetrization of the collision operator is required to analyse the global balance of momentum and energy. We can write [1, 2]:

$$\begin{aligned} C_{\alpha\beta}(\psi_\alpha) + C_{\beta\alpha}(\psi_\beta) \quad &= \quad \sigma_{\alpha\beta}^2 \int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \Big[\Big(\psi_\alpha(\mathbf{c}_\alpha') - \psi_\alpha(\mathbf{c}_\alpha) \Big) f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x}, \mathbf{v}_\beta, \mathbf{x} - \sigma_{\alpha\beta}\mathbf{k}, t) \\ &+ \quad \Big(\psi_\beta(\mathbf{c}_\beta') - \psi_\beta(\mathbf{c}_\beta) \Big) f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x} + \sigma_{\alpha\beta}\mathbf{k}, \mathbf{v}_\beta, \mathbf{x}, t) \Big] (\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\ d\mathbf{k}\ d\mathbf{v}_\alpha\, d\mathbf{v}_\beta \end{aligned} \tag{8}$$

Moreoever, the radial distribution function $\chi_{\alpha\beta}$ for two particles α and β that are respectively centered in $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ is assumed to be equal to the radial distribution function $\chi_{\beta\alpha}$ for two particles β and α that are respectively centered in $\mathbf{x}_\beta$ and $\mathbf{x}_\alpha$, that is:

$$\chi_{\alpha\beta}(\mathbf{x}_\alpha, \mathbf{x}_\beta) = \chi_{\beta\alpha}(\mathbf{x}_\beta, \mathbf{x}_\alpha) \tag{9}$$

The particle diameters σ_α ($\alpha = 1, .., N$) are assumed to be small compared to the characteristic length L of the flow, which measures $\mathbf{x}$. Consequently, Taylor expansions allow to write:

$$\begin{aligned} f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x} + \sigma_{\alpha\beta}\mathbf{k}, \mathbf{v}_\beta, \mathbf{x}, t) \quad &= \quad \sigma_{\alpha\beta}\, k_i \frac{\partial}{\partial x_i} \left[1 - \frac{1}{2}\sigma_{\alpha\beta}\, k_j \frac{\partial}{\partial x_j} + \frac{1}{6}\sigma_{\alpha\beta}^2\, k_j k_m \frac{\partial^2}{\partial x_j \partial x_m} \right] f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x} + \sigma_{\alpha\beta}\mathbf{k}, \mathbf{v}_\beta, \mathbf{x}, t) \\ &+ \quad f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x}, \mathbf{v}_\beta, \mathbf{x} - \sigma_{\alpha\beta}\mathbf{k}, t) + O(\sigma^4) \end{aligned} \tag{10}$$

where σ is the order of magnitude of the diameters of the particles ($\sigma << L$). Moreover, if the definition (4) and the property (9) are used, the following relation is easy to prove:

$$f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x} + \sigma_{\alpha\beta}\mathbf{k}, \mathbf{v}_\beta, \mathbf{x}, t) = f_{\beta\alpha}^{(2)}(\mathbf{v}_\beta, \mathbf{x}, \mathbf{v}_\alpha, \mathbf{x} + \sigma_{\alpha\beta}\mathbf{k}, t) \tag{11}$$

The previous equalities allow to separate the contribution of the collisions in the balance equations in a flux term and in a source term. This leads to the following expression of the right hand side of the global transport equation [1]:

$$\sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \int J_{\alpha\beta}(\mathbf{x}, t)\, \psi_\alpha(\mathbf{c}_\alpha)\, d\mathbf{v}_\alpha = \lambda(\psi) + \frac{\partial}{\partial x_i}\theta_i(\psi) + \theta_i(\frac{\partial \psi}{\partial x_i}) \tag{12}$$

with:

$$\lambda(\psi) = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \frac{\sigma_{\alpha\beta}^2}{2} \int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \Delta\Psi_{\alpha\beta}\, f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x}, \mathbf{v}_\beta, \mathbf{x} - \sigma_{\alpha\beta}\mathbf{k}, t)(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\, d\mathbf{v}_\alpha\, d\mathbf{v}_\beta\, d\mathbf{k} \tag{13}$$

$$\theta_i(\psi) = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \frac{\sigma_{\alpha\beta}^3}{2} \int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \left(\psi_\beta(\mathbf{c}_\beta^{'}) - \psi_\beta(\mathbf{c}_\beta)\right) k_i \left\{1 - \frac{\sigma_{\alpha\beta} k_j}{2}\frac{\partial}{\partial x_j} + \frac{\sigma_{\alpha\beta}^2 k_j k_m}{6}\frac{\partial^2}{\partial x_j \partial x_m}\right\}$$
$$f_{\beta\alpha}^{(2)}(\mathbf{v}_\beta, \mathbf{x}, \mathbf{v}_\alpha, \mathbf{x}+\sigma_{\alpha\beta}\mathbf{k}, t)(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\, d\mathbf{v}_\alpha\, d\mathbf{v}_\beta\, d\mathbf{k} + O(\sigma^6) \quad (14)$$

with $\Delta\Psi_{\alpha\beta} = \psi_\alpha(\mathbf{c}_\alpha^{'}) - \psi_\alpha(\mathbf{c}_\alpha) + \psi_\beta(\mathbf{c}_\beta^{'}) - \psi_\beta(\mathbf{c}_\beta)$. According to this expression, it's easy to observe that, because of the inelasticity of the collisions, energy is not conserved. Moreover, the source term of the momentum balance equation, $\lambda(m\mathbf{c})$, is exactly equal to zero.

Following Grad [4], we expand the single particle distribution function of each species:

$$f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) = f_\alpha = \left[1 + \frac{a_{\alpha ij}}{2T_\alpha^2} c_{\alpha i} c_{\alpha j} - \frac{a_{\alpha imm} c_{\alpha i}}{10 T_\alpha^2}\left(5 - \frac{c_\alpha^2}{T_\alpha}\right)\right] f_{\alpha o}(\mathbf{v}_\alpha, \mathbf{x}, t) \quad (15)$$

where $c_\alpha = \mid \mathbf{c}_\alpha \mid$ and $T_\alpha = < c_\alpha^2 >_\alpha /3$. Now, we use an H-theorem proved in the limit of punctual particles and of a very small inelasticity [2]. In the corresponding thermodynamical equilibrium state, a unique velocity $\mathbf{u}$ and a unique temperature T for all the species are found. We propose a model where all the species have the same mean velocity but different temperatures. Consequently, we set: $\mathbf{u}_\alpha = \mathbf{u}$ and $T_\alpha = T(1+\tau_\alpha)$ for $\alpha = 1,..,N$ in the previous expansion and we set:

$$f_{\alpha o}(\mathbf{v}_\alpha, \mathbf{x}, t) = n_\alpha \left(\frac{1}{2\pi T_\alpha}\right)^{\frac{3}{2}} exp\left[\frac{-(\mathbf{v}_\alpha - \mathbf{u})^2}{2T_\alpha}\right] \quad (16)$$

with: $n = \sum_{\alpha=1}^{N} n_\alpha$. This Grad expansion can be interpreted as an expansion around a Maxwellian state of equilibrium which caracterizes punctual particles undergoing elastic collisions. The effects of the inelasticty and of the size of the particles are considered as perturbations around this equilibrium state. As in the usual homogeneous case, the thirteen moments $a_{\alpha i}$, $a_{\alpha ij}$, $a_{\alpha imm}$ ... depend only on the position $\mathbf{x}$ and on time t. Moreover, it is very easy to show that $a_{\alpha i} = 0$ and $a_{\alpha ij} = a_{\alpha ji}$. The proof of these results is given in the case of an homogeneous suspension in [5] for instance. The generalization to heterogeneous dispersed media is obvious. In the frame of the Standard Enskog Theory, approximated formulations for the radial distribution function $\chi_{\alpha\beta}$ can be used [14]. It is evaluated as a function of the local densities n_δ ($\delta = 1,..,N$) at a point $\mathbf{X}_{\alpha\beta}$ that we set as $\mathbf{X}_{\alpha\beta} = \mathbf{x} + \mu_{\alpha\beta}\,\sigma_{\alpha\beta}\mathbf{k}$ where the parameter $\mu_{\alpha\beta}$ is a real number ($0 \le \mu_{\alpha\beta} \le 1$). Naturally, the choice of this parameter allows to give to $\mathbf{X}_{\alpha\beta}$ various definitions, as those that were introduced in the first section. For instance, if $\mu_{\alpha\beta} = 1/2$ then $\mathbf{X}_{\alpha\beta}$ is the midpoint of the line joining the centers of the two colliding spheres. Nevertheless, at the same accuracy level as in [5], a symmetrization of the source term (13) of the collision operator allow to build a model that does not depends on the value of $\mu_{\alpha\beta}$ [3]. In order to build such a model, we set:

$$\chi_{\alpha\beta}(\mathbf{x}, \mathbf{x} \pm \sigma_{\alpha\beta}\mathbf{k}) = \chi_{\alpha\beta}(\mathbf{x} \pm \mu_{\alpha\beta}\,\sigma_{\alpha\beta}\mathbf{k}) \quad (17)$$

We will show in the following, that the terms depending on $\mu_{\alpha\beta}$ disappear of the equations at the chosen level of accuracy. Then, the two particle distribution function can be written as:

$$f_{\alpha\beta}^{(2)}(\mathbf{v}_\alpha, \mathbf{x}, \mathbf{v}_\beta, \mathbf{x} - \sigma_{\alpha\beta}\mathbf{k}, t) = \chi_{\alpha\beta}(\mathbf{x} - \mu_{\alpha\beta}\,\sigma_{\alpha\beta}\mathbf{k})\, f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)\, f_\beta(\mathbf{v}_\beta, \mathbf{x} - \sigma_{\alpha\beta}\mathbf{k}, t) \quad (18)$$

With this hypothesis about the radial distribution function, the expansion of the single particle distribution function near a Maxwellian state, and consequently the derivation of the collision terms of the balance equations, is simplified. Following [3], the source term (13) is expressed as:

$$\lambda(\psi) = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \mathscr{L}_{\alpha\beta} \quad (19)$$

with:

$$\mathscr{L}_{\alpha\beta} = \frac{\sigma_{\alpha\beta}^3}{4}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \Delta\Psi_{\alpha\beta}\,\chi_{\alpha\beta}(\mathbf{x})\, f_\alpha f_\beta \frac{\sigma_{\alpha\beta}}{2} k_l \frac{\partial}{\partial x_l}\left(ln\frac{f_\alpha}{f_\beta}\right)(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\, d\mathbf{v}_\alpha\, d\mathbf{v}_\beta\, d\mathbf{k}$$
$$+ \frac{\sigma_{\alpha\beta}^2}{2}\int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \Delta\Psi_{\alpha\beta}\,\chi_{\alpha\beta}(\mathbf{x})\, f_\alpha f_\beta (\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})\, d\mathbf{v}_\alpha\, d\mathbf{v}_\beta\, d\mathbf{k} + O(\sigma^4) \quad (20)$$

where $f_\alpha = f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)$ and $f_\beta = f_\beta(\mathbf{v}_\beta, \mathbf{x}, t)$. In order to obtain a description of the same level of accuracy as the description introduced in the paper of Jenkins *et al.* [5] for a homogeneous mixture, we keep only the terms of the two lowest order in the previous relation and we obtain, as in [3], an expression of the source term (20) that does not depend on $\mu_{\alpha\beta}$. In [5], the authors make two additionnal assumptions regarding the flow. First, they consider that the spatial derivatives of the mean field are small and then the flows of dense suspension with large mean velocity gradient are out of the scope of their model. Second they focuse on flows in which the departure of the distribution function from the Maxwellian state is small. The same assumptions are made in this work for the heterogeneous mixture: We assume that the quantities a_{ij}/T, τ_α and $a_{ijk}/T^{3/2}$ are small and of the same order of magnitude. With these assumptions, we write an approximate expression for the collisional source and the collisional flux that are linear in the perturbations a_{ij}, τ and a_{ijk} and the spatial gradients. Consequently, the collisional flux term (14) may be written as:

$$\theta_i(\psi) = \sum_{\alpha,\beta=1}^{N} \theta_{\alpha\beta\, i} = \sum_{\alpha,\beta=1}^{N} \frac{\sigma_{\alpha\beta}^3}{2} \chi_{\alpha\beta}(\mathbf{x}) \int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} \left(\psi_\beta(\mathbf{c}_\beta^{'}) - \psi_\beta(\mathbf{c}_\beta)\right) k_i f_\alpha f_\beta \, d\mathbf{v}_\alpha \, d\mathbf{v}_\beta \, d\mathbf{k} + O(\sigma^4) \tag{21}$$

where only the lowest order is retained in order to write an expression of the collisional flux at the same order of precision as $\mathscr{L}_{\alpha\beta}$. According to the case of the source term, as in [3], we remark that $\mu_{\alpha\beta}$ does not appear in the previous expression at this order of approximation. Then, the integrations are carried out with the help of a formal calculation software. In order to be intelligible, the details of the calculations are not given in this paper: they are standard (see for instance [10], [5]) but too long to be reproduced here. Finally, we obtain the expressions for the collision terms of the global balance equations. We give, for instance, the flux term of the global momentum balance equation:

$$\theta_{\alpha\beta\, i}(m_\beta\, c_{\beta\, j}) = -\frac{2\pi\,(1+e_{\alpha\beta})\,\rho_\alpha\,\rho_\beta}{15(m_\alpha + m_\beta)} \sigma_{\alpha\beta}^3\, \chi_{\alpha\beta} \left[5\,T\,\delta_{i\,j}\left(1 + \frac{\tau_\alpha + \tau_\beta}{2}\right) + a_{\alpha\, i\, j} + a_{\beta\, i\, j}\right] + O(\sigma_{\alpha\beta}^4) \tag{22}$$

All the other terms have been explicitly calculated but they are not reproduced here for the sake of brevity. The results are consistent with the homogeneous case and with the case where all the species have the same temperature. Compared to this last case, the temperature difference between the species generates a source of energy and a flux of third order moments; that is only the source term of the energy balance and the flux term of the third order moments balance equation are modified.

TRANSPORT EQUATIONS FOR EACH SPECIES

Now, we will derive the balance equations for each species. This has already be done in the case of the mass conservation. For the other quantities (momentum, energy and triple velocity correlations), we start from the general balance equation for the species α (6). An explicit form of the left hand side member of this equation has been given in [2]. Obviously, the aim of this section is to obtain a set of equations at the same accuracy level as the global balance set of equations presented in the previous section. Consequently, we are still in the frame of the Standard Enskog Theory and then we assume that the two particle distribution function can be expressed with the relation (18). As in the previous section, we assume that the particle diameters are small compared to the characteristic length of the flow. Then, as in [3], the collision term of the general balance equation for the species α can be expressed as follows:

$$C_{\alpha\beta}(\psi_\alpha) = \gamma_{\alpha\beta}(\psi_\alpha) - \frac{\partial}{\partial x_j}\theta_{\alpha\beta\, j}(\psi_\alpha) - \frac{\partial u_{\alpha l}}{\partial x_j}\theta_{\alpha\beta\, j}(\frac{\partial \psi_\alpha}{\partial c_{\alpha l}}) + O(\sigma_{\alpha\beta}^4) \tag{23}$$

with :

$$\gamma_{\alpha\beta}(\psi_\alpha) \;=\; \frac{\sigma_{\alpha\beta}^2}{2} \int_{(\mathbf{g}_{\beta\alpha}\cdot\mathbf{k})>0} (\mathbf{g}_{\beta\alpha}\cdot\mathbf{k}) \left(\psi_\alpha^{'} - \psi_\alpha\right) \chi_{\alpha\beta} \left[2 f_\alpha f_\beta + \sigma_{\alpha\beta} k_j \left(f_\beta \frac{\partial f_\alpha}{\partial x_j} - f_\alpha \frac{\partial f_\beta}{\partial x_j}\right)\right] d\mathbf{k}\, d\mathbf{v}_\beta\, d\mathbf{v}_\alpha \tag{24}$$

where $\psi_\alpha = \psi_\alpha(\mathbf{c}_\alpha)$ and $\psi_\alpha^{'} = \psi_\alpha(\mathbf{c}_\alpha^{'})$. At this approximation order, the collision term does not depend on the value of the coefficient $\mu_{\alpha\beta}$ which appears in the definition of $\chi_{\alpha\beta}$ in the Standard Enskog Theory. Consequently, at the same accuracy level as in the case of the global balance equations, we obtain, for each species, a set of balance equations that do not depend on the value of $\mu_{\alpha\beta}$. In this expression, the terms in $\theta_{\alpha\beta j}$ are easy to calculate from the flux terms given in the previous section. On the other hand, the source terms $\gamma_{\alpha\beta}(\psi_\alpha)$ are unknown and must be evaluated. As for the global balance equations, the end of the calculations is standard but not very easy: It's too long to be reproduced here. Finally, we obtain the expressions for this contribution to the collision terms of the balance equations for each species. As in the previous section, all the terms have been explicitly calculated but, for the sake of brevity, we give the final result only for the energy of the species α:

$$\begin{aligned}
\gamma_{\alpha\beta}(m_\alpha c_{\alpha i} c_{\alpha j}) &= \frac{8(1+e_{\alpha\beta})\rho_\alpha \rho_\beta}{15(m_\alpha+m_\beta)} \chi_{\alpha\beta} \sigma_{\alpha\beta}^2 \sqrt{\pi T} \left[5(a_{\beta ij} - a_{\alpha ij}) + 5T(\tau_\beta - \tau_\alpha)\delta_{ij} \right. \\
&+ \left. 10 \frac{e_{\alpha\beta} m_\beta - m_\alpha}{m_\alpha + m_\beta} T \left(1 + \frac{3}{4}(\tau_\alpha + \tau_\beta)\right)\delta_{ij} + 3\frac{m_\beta(e_{\alpha\beta}-1) - 2m_\alpha}{m_\alpha+m_\beta}(a_{\alpha ij} + a_{\beta ij}) \right] \\
&+ \frac{2(1+e_{\alpha\beta})\rho_\alpha \rho_\beta}{15(m_\alpha+m_\beta)} \chi_{\alpha\beta} \sigma_{\alpha\beta}^3 \pi T \left[\frac{9m_\alpha + 3m_\beta(1-2e_{\alpha\beta})}{m_\alpha+m_\beta} \left(\frac{\partial u_{\alpha i}}{\partial x_j} + \frac{\partial u_{\alpha j}}{\partial x_i} \right) \right. \\
&+ \left. \frac{4m_\alpha - 2m_\beta(1+3e_{\alpha\beta})}{m_\alpha+m_\beta} \frac{\partial u_{\alpha k}}{\partial x_k} \delta_{ij} \right] + O(\sigma_{\alpha\beta}^4)
\end{aligned} \tag{25}$$

The results are consistent with the case where all the species have the same temperature [3]. Compared to this case, the source terms of the momentum balance equation and of the third order moments balance equation are unchanged: Only the source term of the energy balance equation is modified.

CONCLUSION

An Eulerian description for a heterogeneous suspension constituted of N different solid particle species with its own temperature is introduced from Grad's thirteen moment method in the frame of the Standard Enskog Theory. The distinction between the temperature of each species bring about some modifications of the collision terms of the balace equations but the method presented in [3] are still suitable: At the usual level of accuracy for a suspension [5], the macroscopic description does not depend on the point where the the radial distribution function is written. In the future, different asymptotic cases for a suspension with two types of particles will be analysed: The case of a great difference in the particle sizes and/or masses and the case of a great difference in the number densities of each species.

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